Mystery of the Aleph

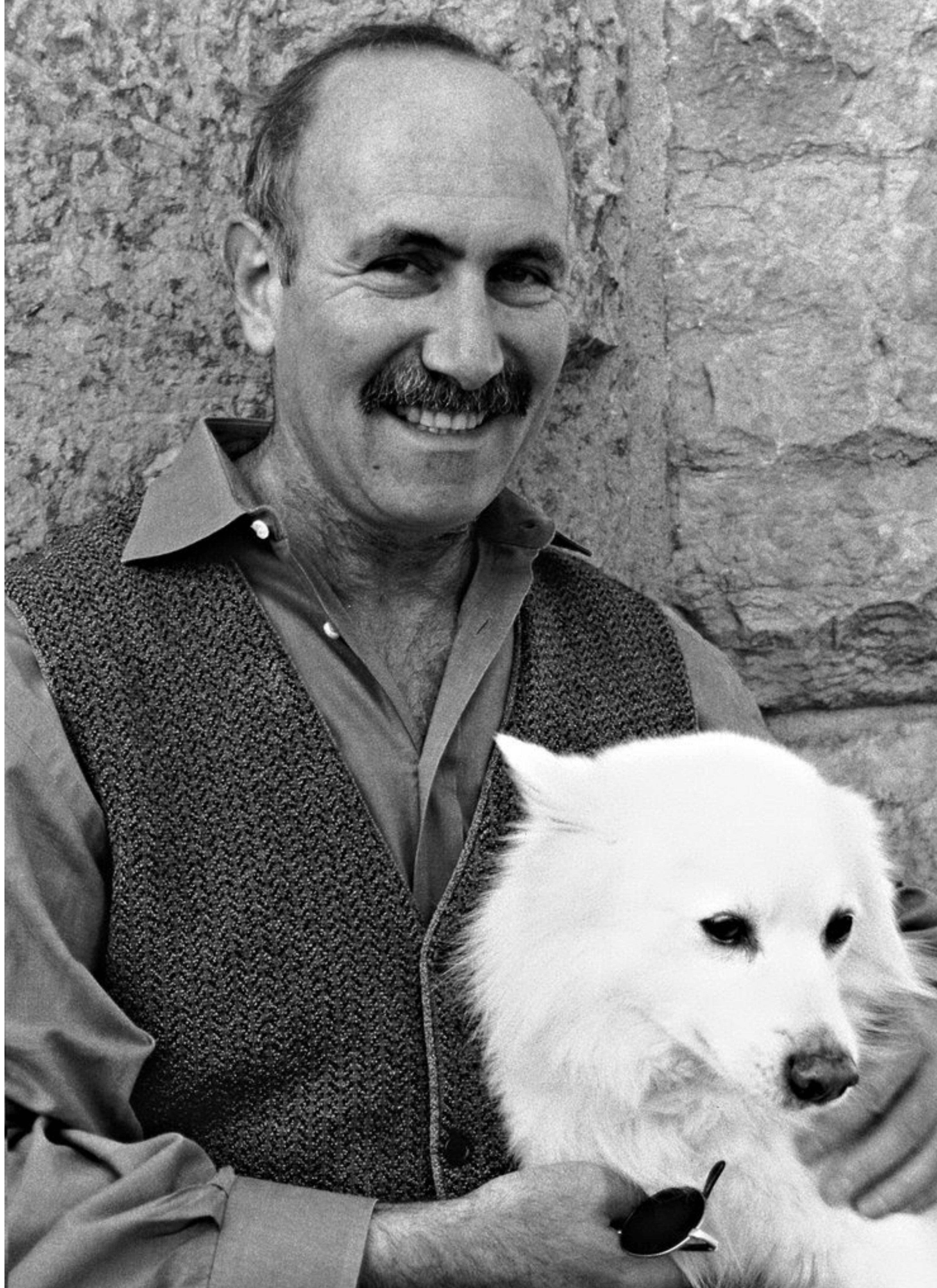
Amir Aczel

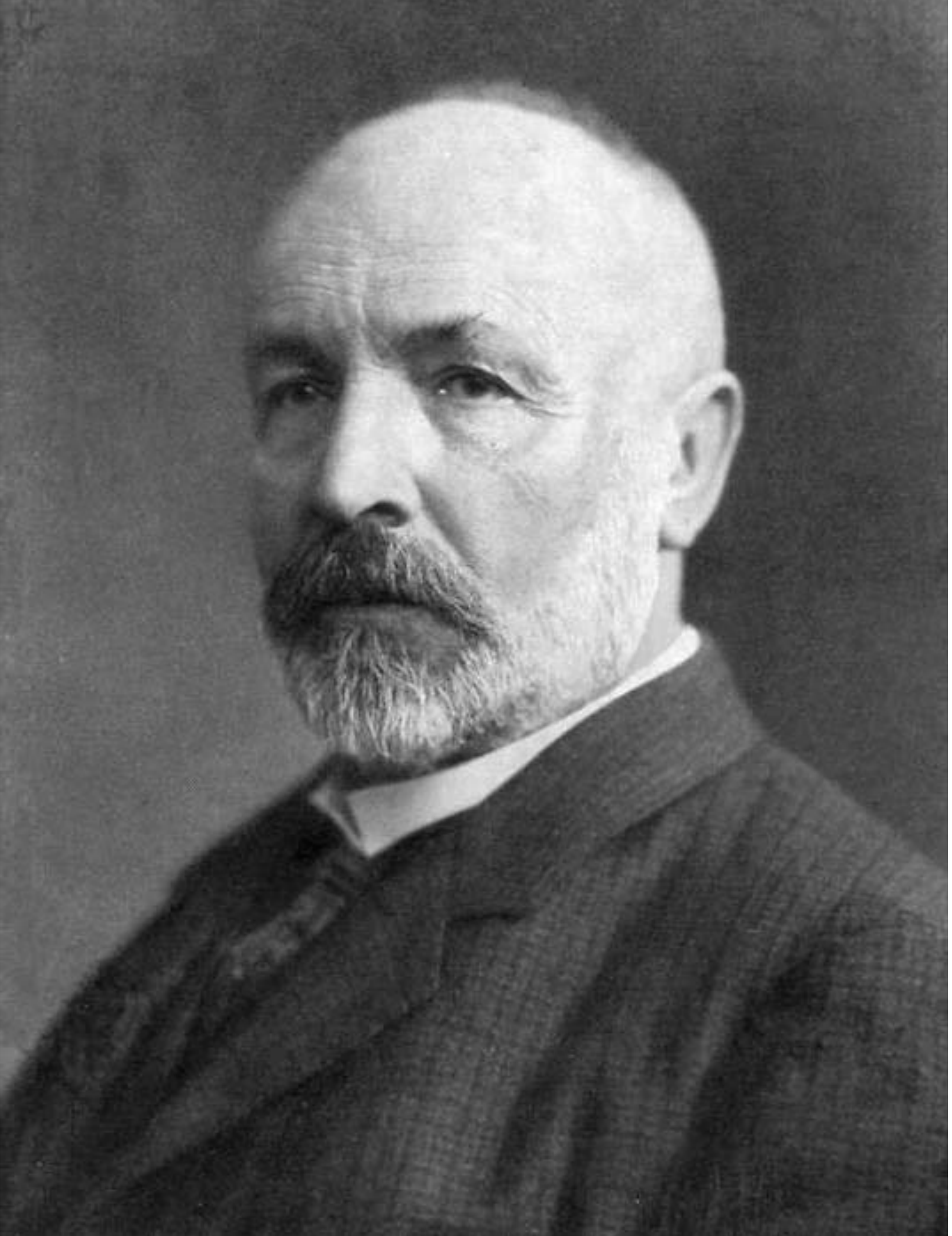
Tessellate book club - 09.01.2024
Presented by Davide Bassetti

Amir Aczel

1950-2015

- Israeli-born American lecturer in mathematics and history of science
- Author of popular science books, including:
  - The Mystery of the Aleph: Mathematics, the Kabbalah, and the Search for Infinity, 2000
  - The Riddle of the Compass: The Invention that Changed the World, 2001
  - Entanglement: The Greatest Mystery in Physics, 2002
  - The Artist and the Mathematician: The Story of Nicolas Bourbaki, the Genius Mathematician Who Never Existed, 2007
  - Finding Zero, 2015





Mystery of the Aleph

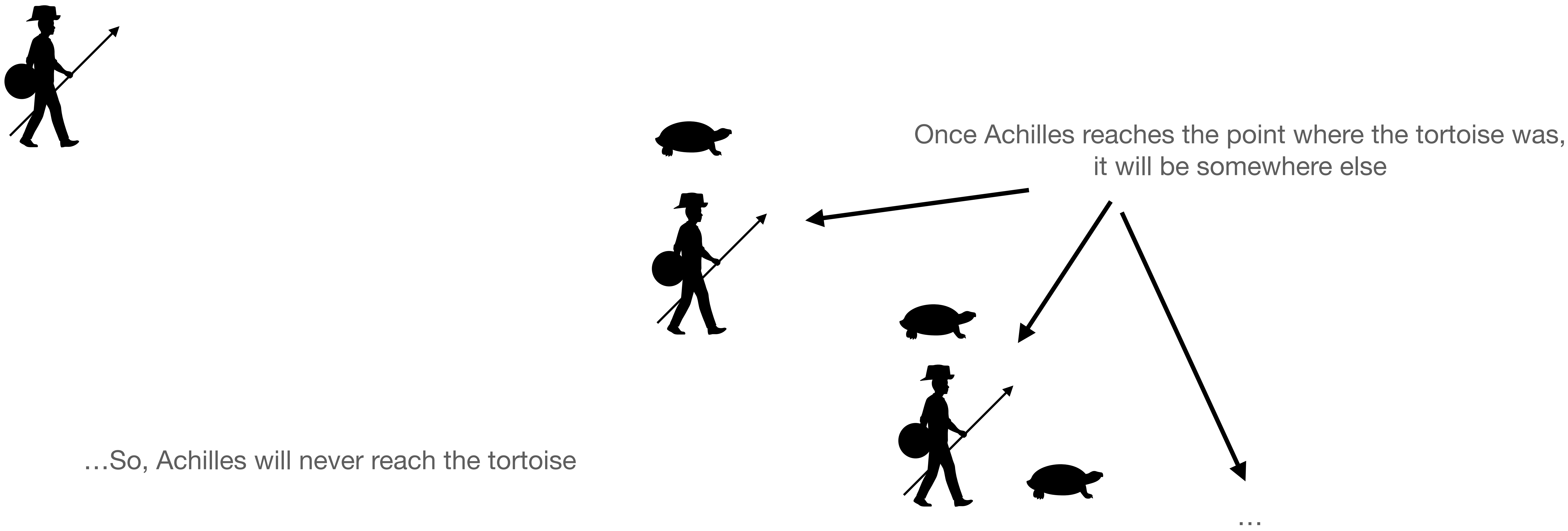
Published in 2000

- Life of Georg Cantor, mathematician
- Born Georg Ferdinand Ludwig Phillipp Cantor in 1845, died in 1918
- Became not-so-happily Professor at Halle, after his PhD in Berlin
- Plagued in his later life by mental unwellness
  - Azcel makes the point that his episodes of depression were associated with a single expression

Ancient roots

Zeno’s paradoxes I

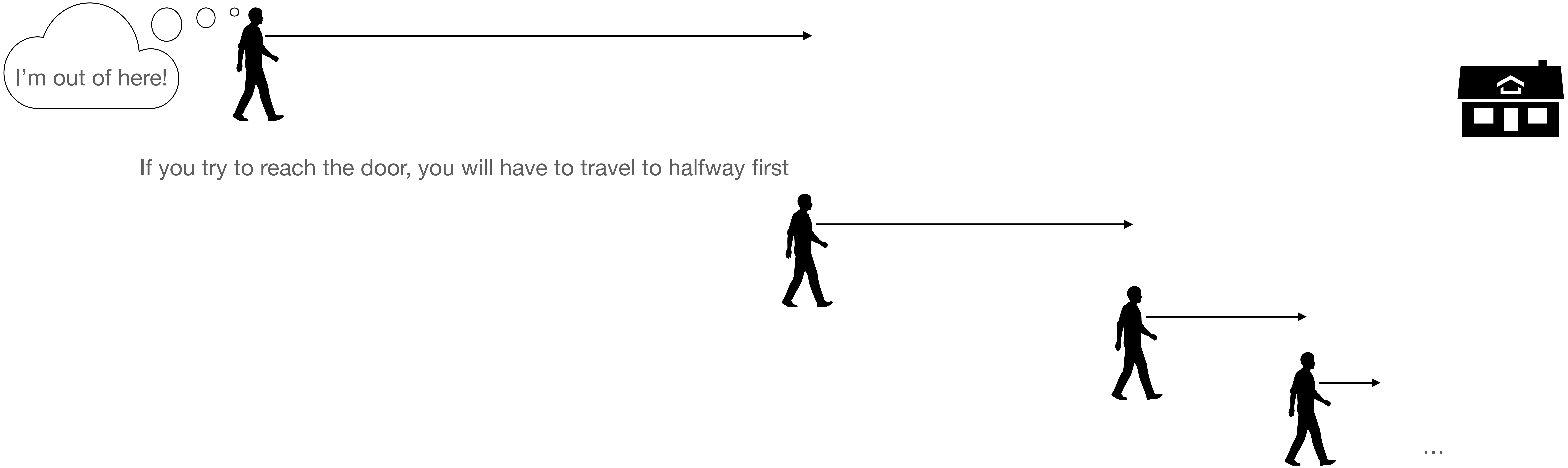
- The greeks first discovered the concept of infinity (causing “at least one murder”)
    - Zeno’s paradoxes (Achilles and the tortoise, the room paradox,…)



Ancient roots

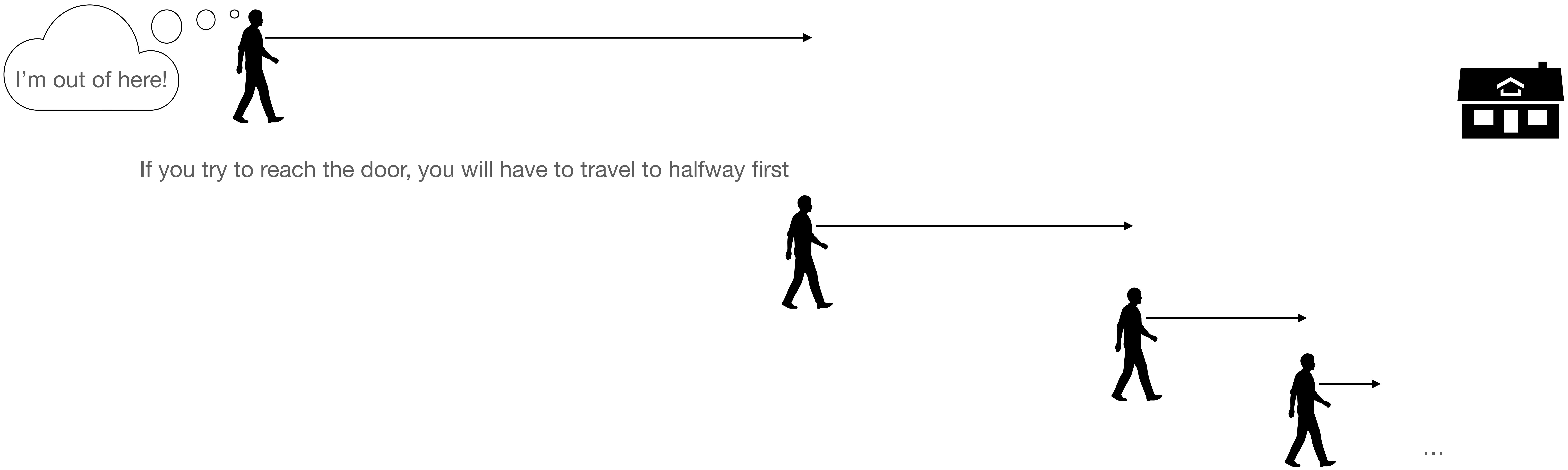
Zeno's paradoxes II

- The greeks first discovered the concept of infinity (causing "at least one murder")
  - Zeno's paradoxes (Achilles and the tortoise, the room paradox,…)



Ancient roots

Zeno's paradoxes III

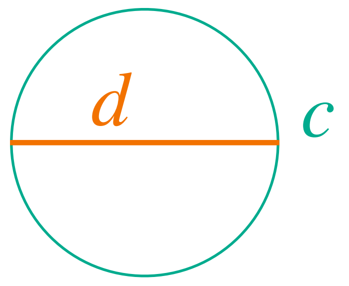


- Infinitely many steps can lead to a finite distance travelled

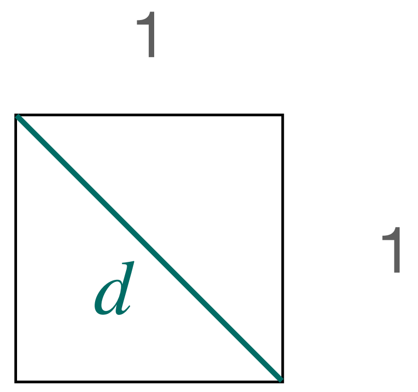
$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2$$

Ancient roots

Pythagoras



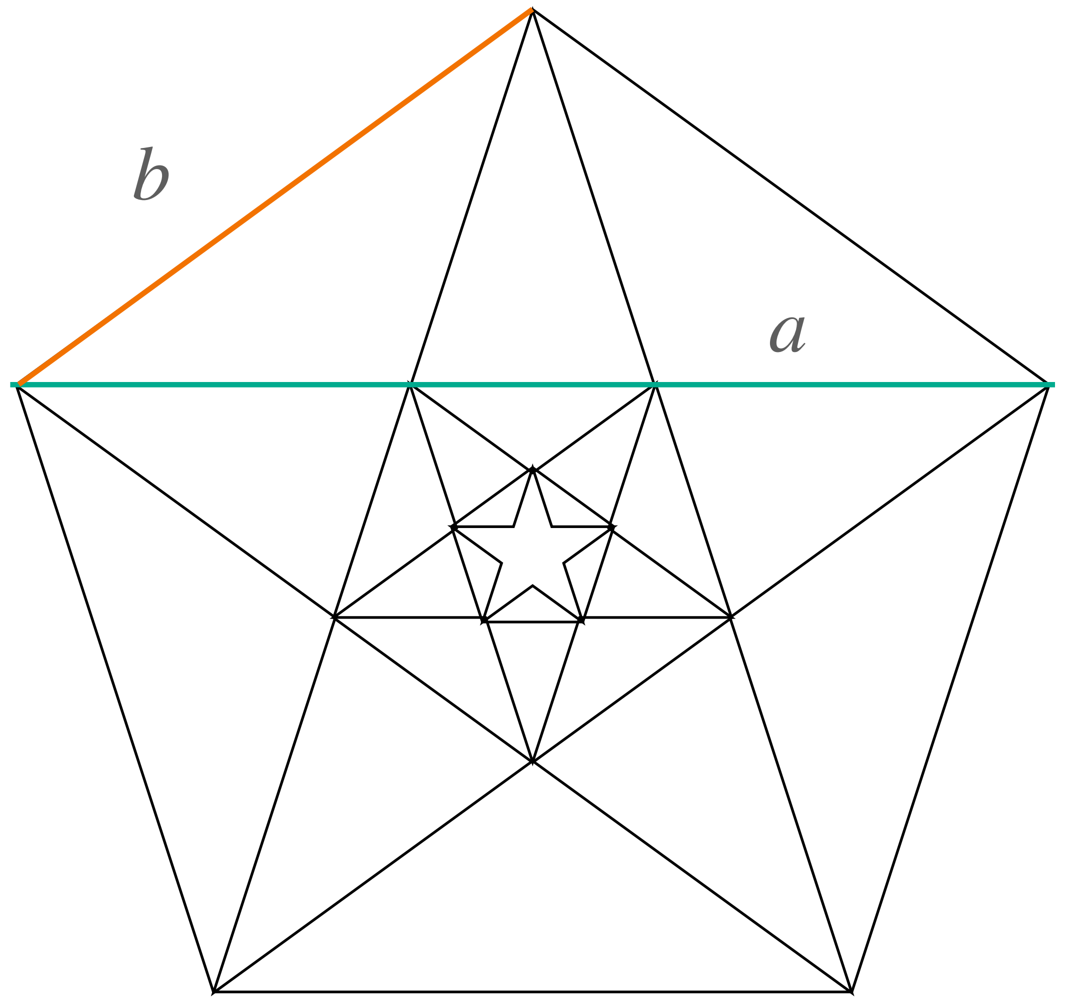
$$\frac{c}{d} = \pi = 3.1415926535\ldots$$



$$d = \sqrt{2} = 1.41421356\ldots$$

- Pythagoras (after travelling to Egypt) founded modern geometry by introducing rigorous mathematics, axioms and logic.
- “God is number” was the pythagorean’s motto.
- 10 is a special number
- He aimed at explaining the world using integer and rationals
- However, they knew about irrational numbers (kept secret)
- Ratio of successive Fibonacci numbers also converges to φ

$$1,1,2,3,5,8,13,21,\ldots$$



$$\frac{a}{b} = \varphi = 1.618033989\ldots$$

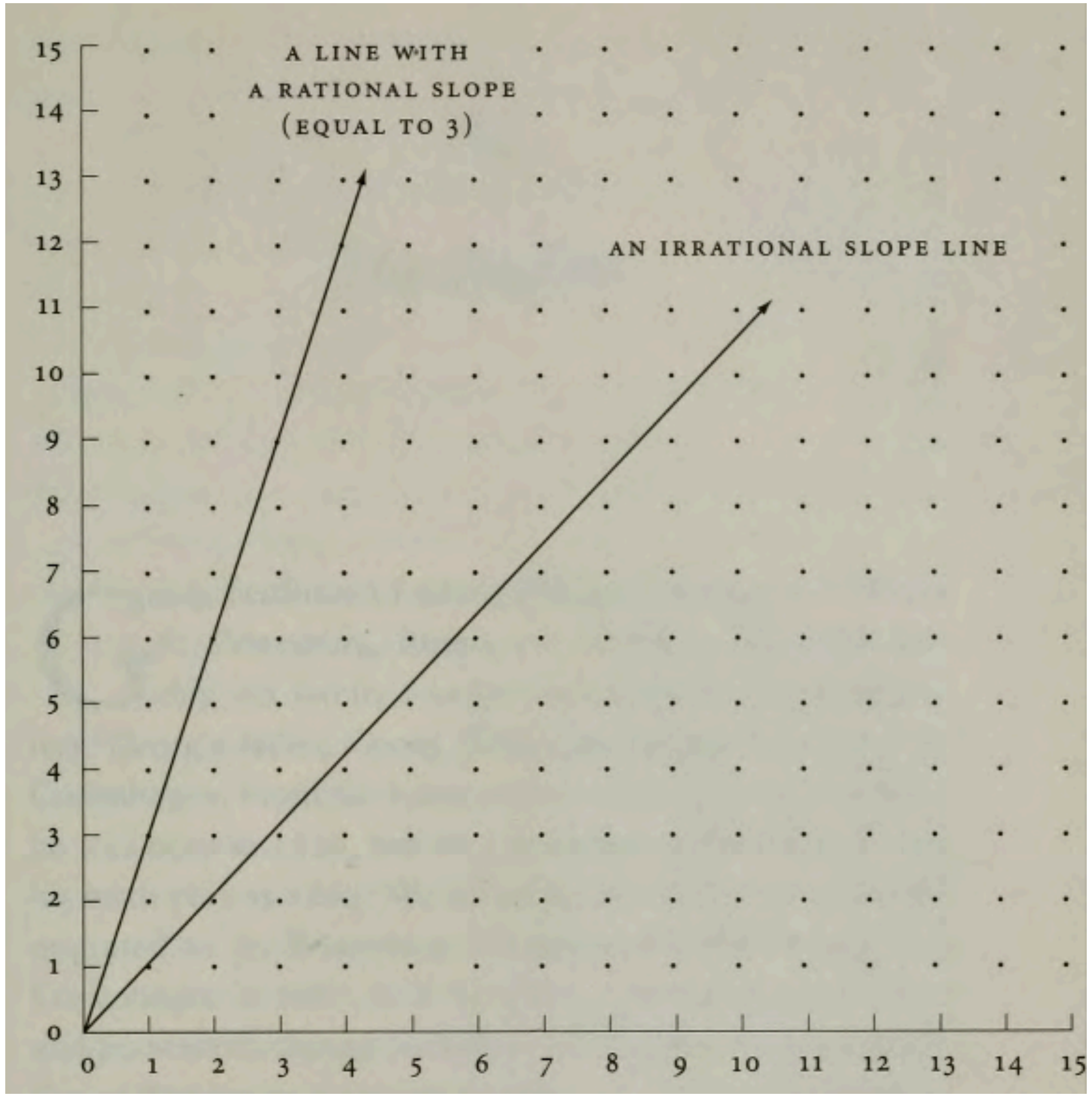
Irrational numbers

- Consider the integers, displayed in 2 dimensions
- If we draw a ray from {0,0}, with a rational slope, we will “hit” a point in the array

$$3 = \frac{3}{1} = \frac{6}{2} = \frac{9}{3} = \dots$$

- A line with an irrational slope does not intersect any point

$$\pi \approx \frac{22}{7}$$



Ancient roots

The method of exhaustion

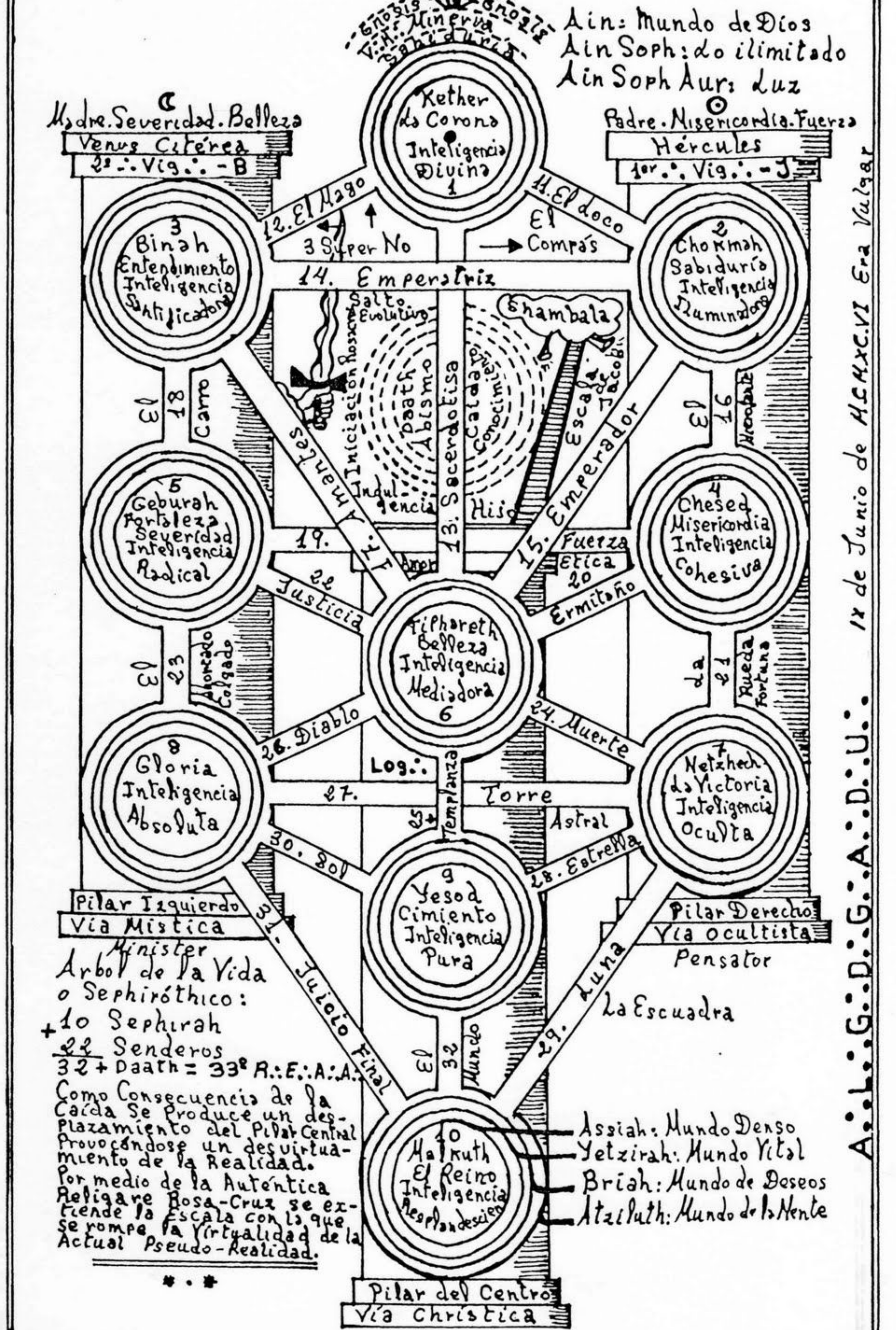
- Eudoxos (Plato’s student) and Archimedes used limit processes to calculate areas, adding many infinitesimally small parts
- Then, for 2000 years, nothing much more…
- Christian perspective was not fond of infinity

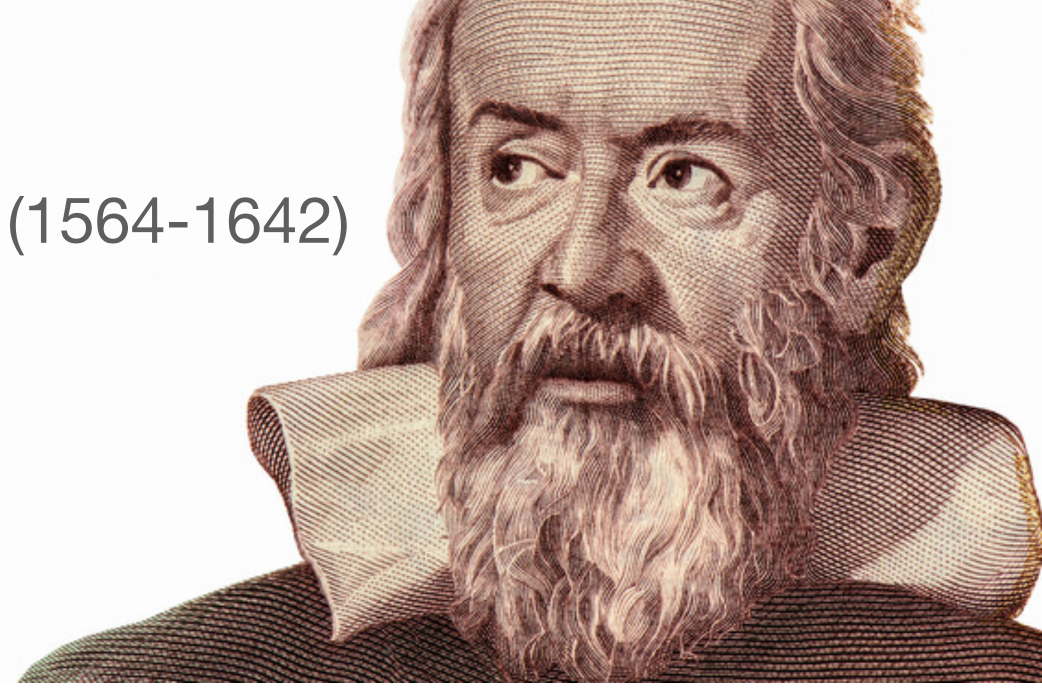
Kabbalah

- Isrealites transmitted orally practices to achieve religious experiences through meditation, which was named Kabbalah
- It survived as a secret tradition, and numbers played an important role
- For example Gematria
  - Each letter is associated with a number, a name has a “sum” and a hidden meaning
- Collected in the Zohar in the 12th century

Kabbalah

- 10 sephirot
- Each of the four letter of the name of god forms a world, and each permutation forms a sephirot
- Each one has a meaning and a colour, and is an aspect of god
- God, however, is much bigger. Its name is *ein sof*, which means “without end”, “interminable”, … or “**infinity**”





Galileo Galilei (1564-1642)

Counting

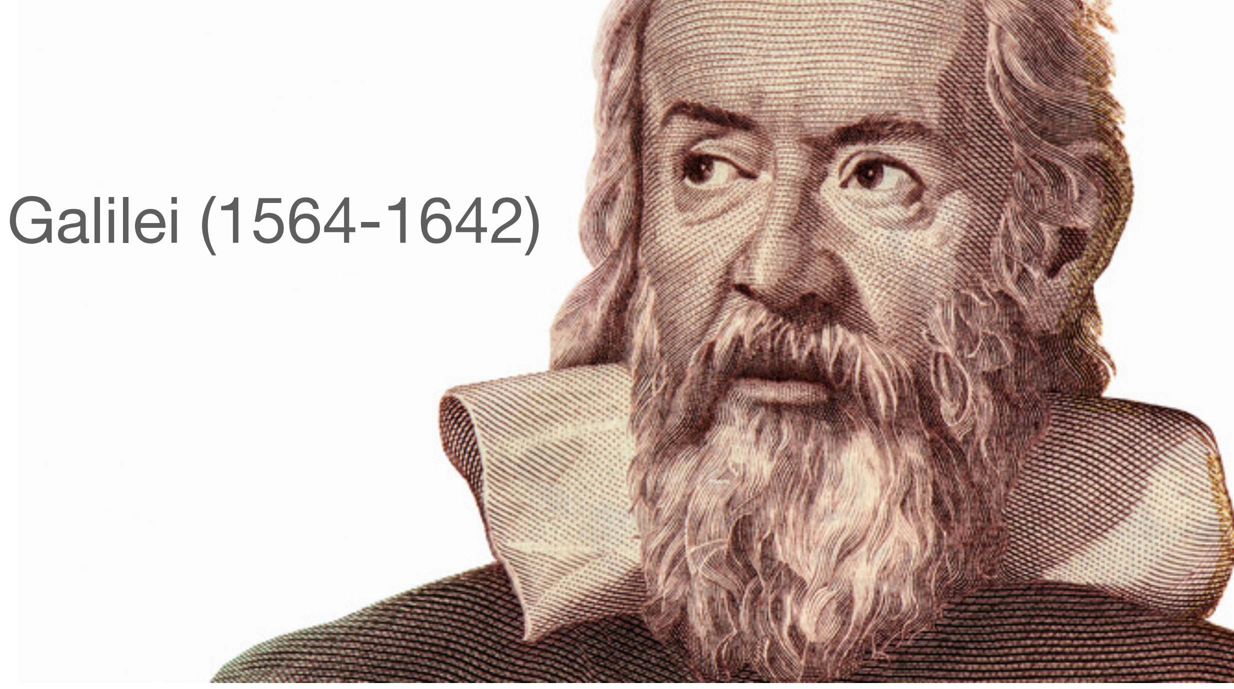
- Let us consider the set of all natural numbers

$\mathbb{N}$

1 2 3 4 5 6 ...

Counting

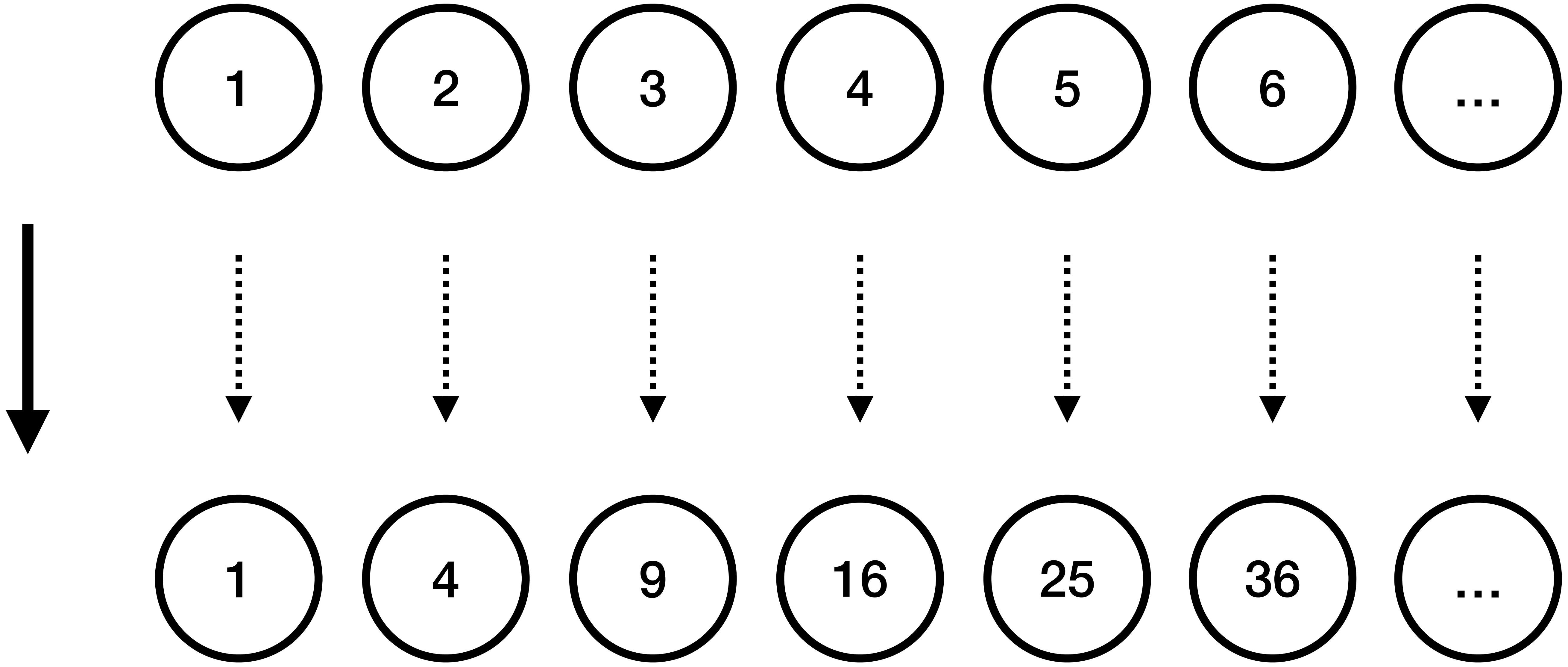
Galileo Galilei (1564-1642)



$\mathbb{N}$

- Same number of elements

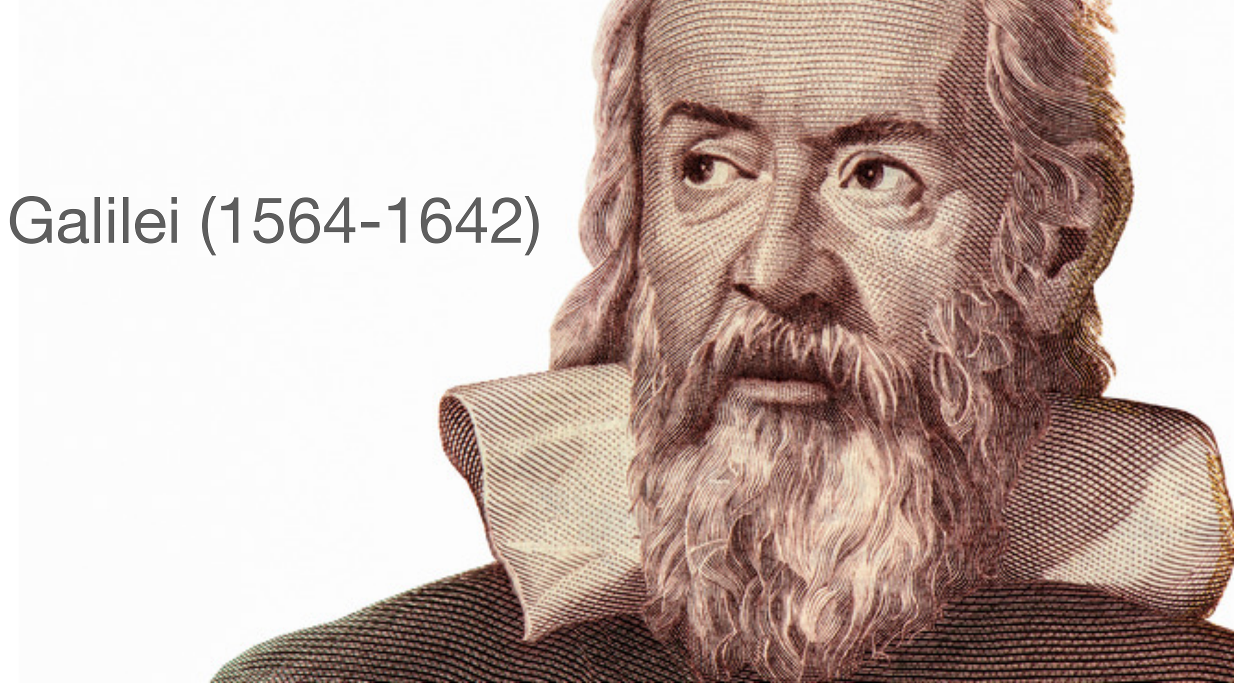
$f(x) = x^2$



Square numbers

- A subset of an infinite set can have the same cardinality of the original set

$|\mathbb{Z}| = |\mathbb{N}|$

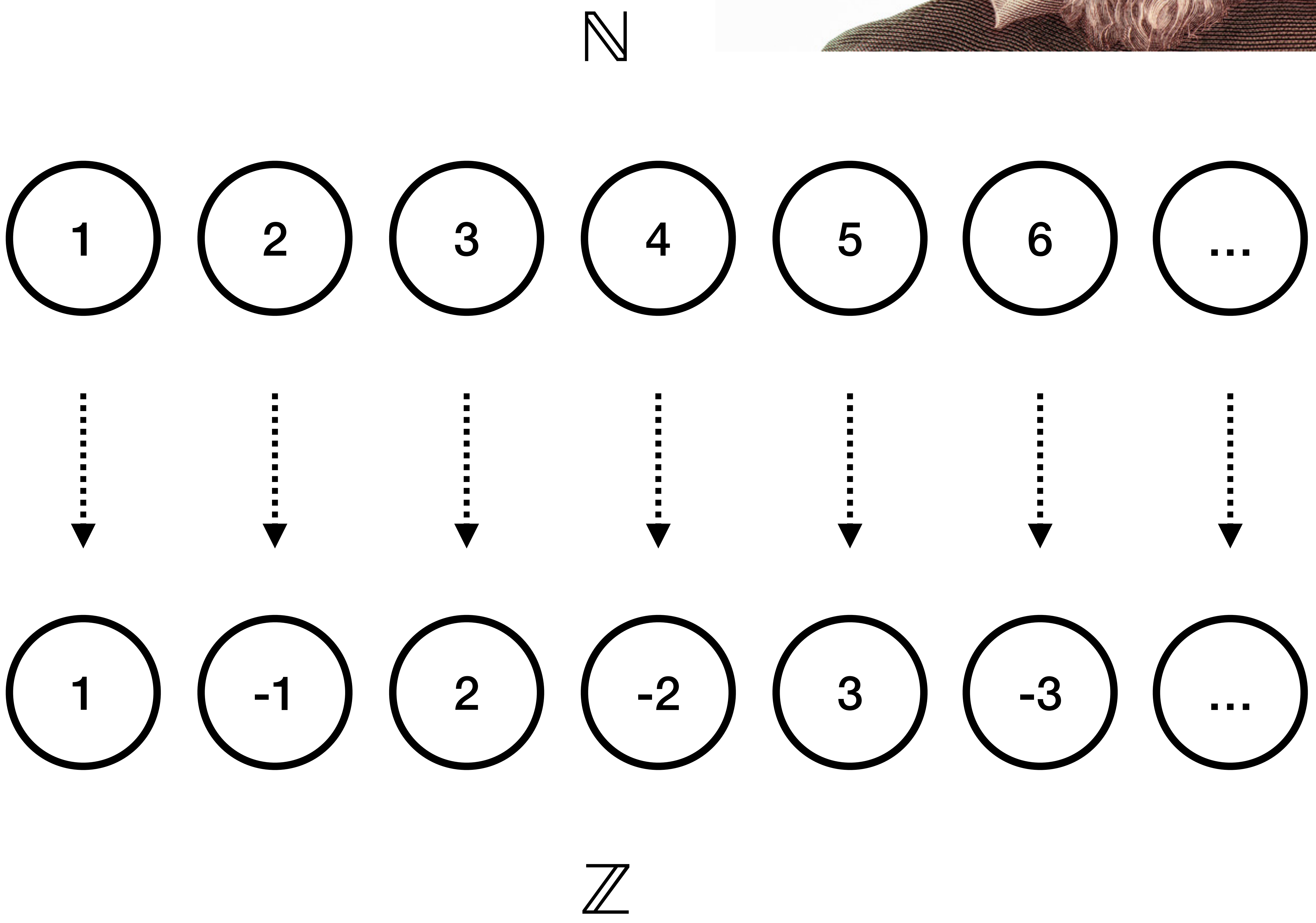


Galileo Galilei (1564-1642)

Counting

- Same number of elements

$\cup \{-1, -2, -3, \ldots\}$



- Adding something countable does not change the cardinality

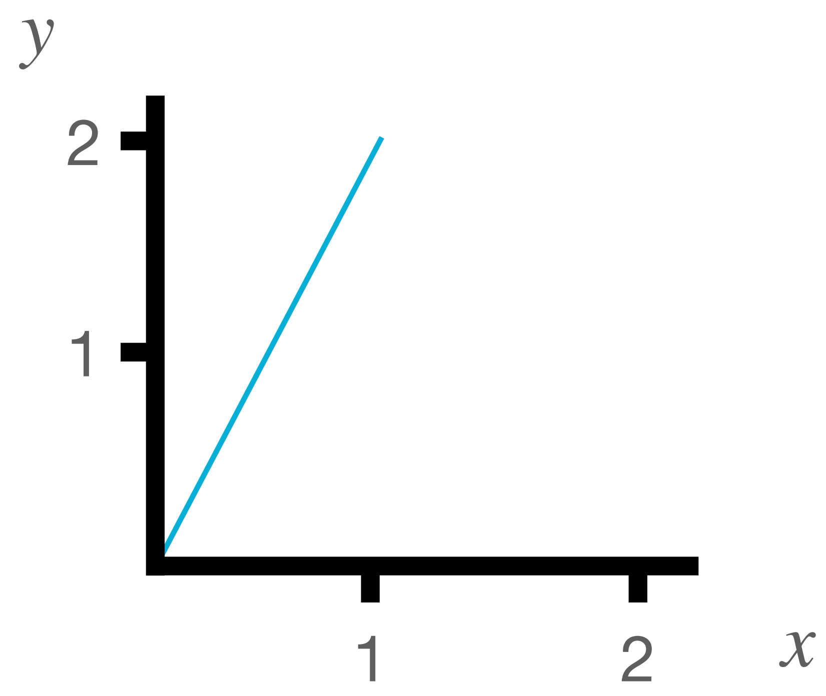
- Countable infinities

$|\mathbb{Z}| = |\mathbb{N}|$

The paradoxes of infinity

Bernard Bolzano (1781-1848)

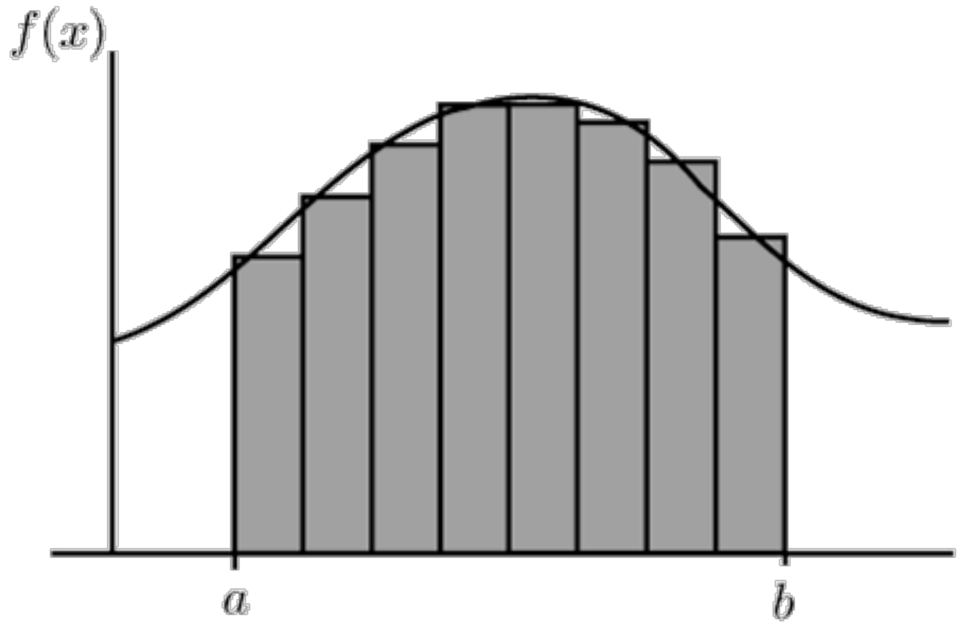
- Similar argument, but on continuum:
  - Map $[0,1]$ to $[0,2]$
  - $y = 2x$



  - A closed interval contains as many numbers as any other closed interval

Infinity and the late 1800s in Berlin

- Bernhard Riemann (1826-1866)
  - Riemann integration



- Karl Weierstraß (1815-1897)
  - Approximating function using infinite series

Infinity and the late 1800s in Berlin

- Leopold Kronecker (1823-1891)
- Studied algebra (discrete elements), and ignored Weierstraß

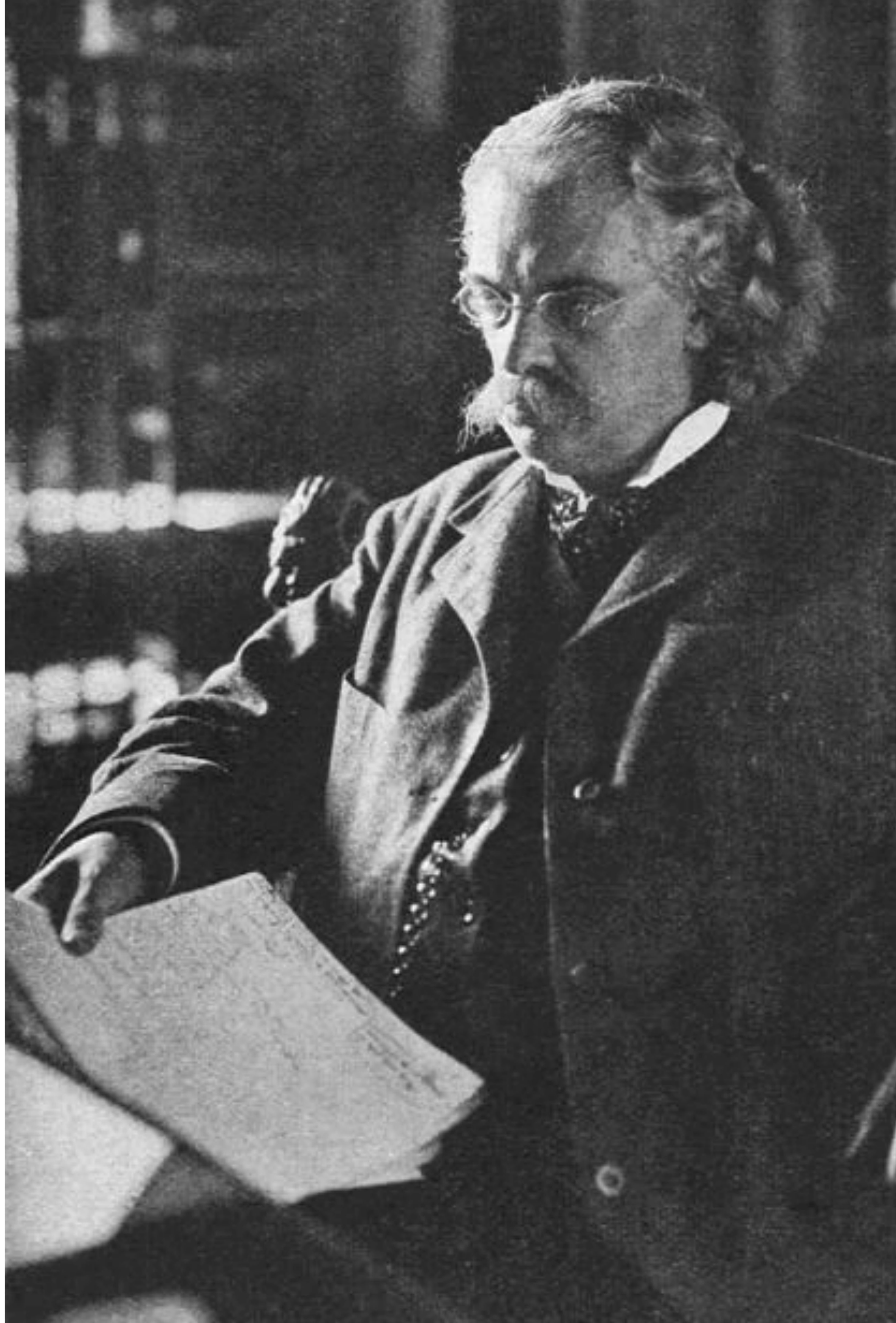
"Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk"

Georg Cantor

- In 1856 his family moved to Frankfurt, where Georg attended private schools, and went to the Gymnasium in Darmstadt
- He then studied in Berlin, where he attended lectures from Weierstraß and Kronecker
- He wrote a dissertation on number theory in 1867, and subsequently became professor in Halle
- He worked was inspired by the methods of Bolzano and Weierstraß, which put him in conflict with Kronecker

Gösta Mittag-Leffler
(1846-1927)

- Swedish student of Weierstraß
- Weierstraß "did not believe in publishing his results and did not even like to have students take lecture notes at his course", many of his work survived thanks to Mittag-Leffler.
- Also, he could be the reason why there is no Nobel prize for Mathematics
- Close friend of Cantor, he would publish his work in his journal *Acta Mathematica* even when nobody would consider his work on infinity



Richard Dedekind

1831-1916

- Another close friend of Cantor
- Working at the small university of Brunswick
- Supported Cantor's ideas
- Anecdote: Cantor did not speak to him for 17 years after Dedekind refused the offer as professor of Halle that Cantor organised for him

- Cantor thought about infinity from a set-theory perspective, after having contributed the theory of sets, based on the ideas of Peano

$$S = \{1,2,3\} \qquad Q = \{Apples, Dogs, Key, Computer\}$$

- Peano devised a number system from sets

$$\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$$

⭐★★★★

- “Cantor’s set theory was a malady, a perverse illness from which some day mathematics would be cured”

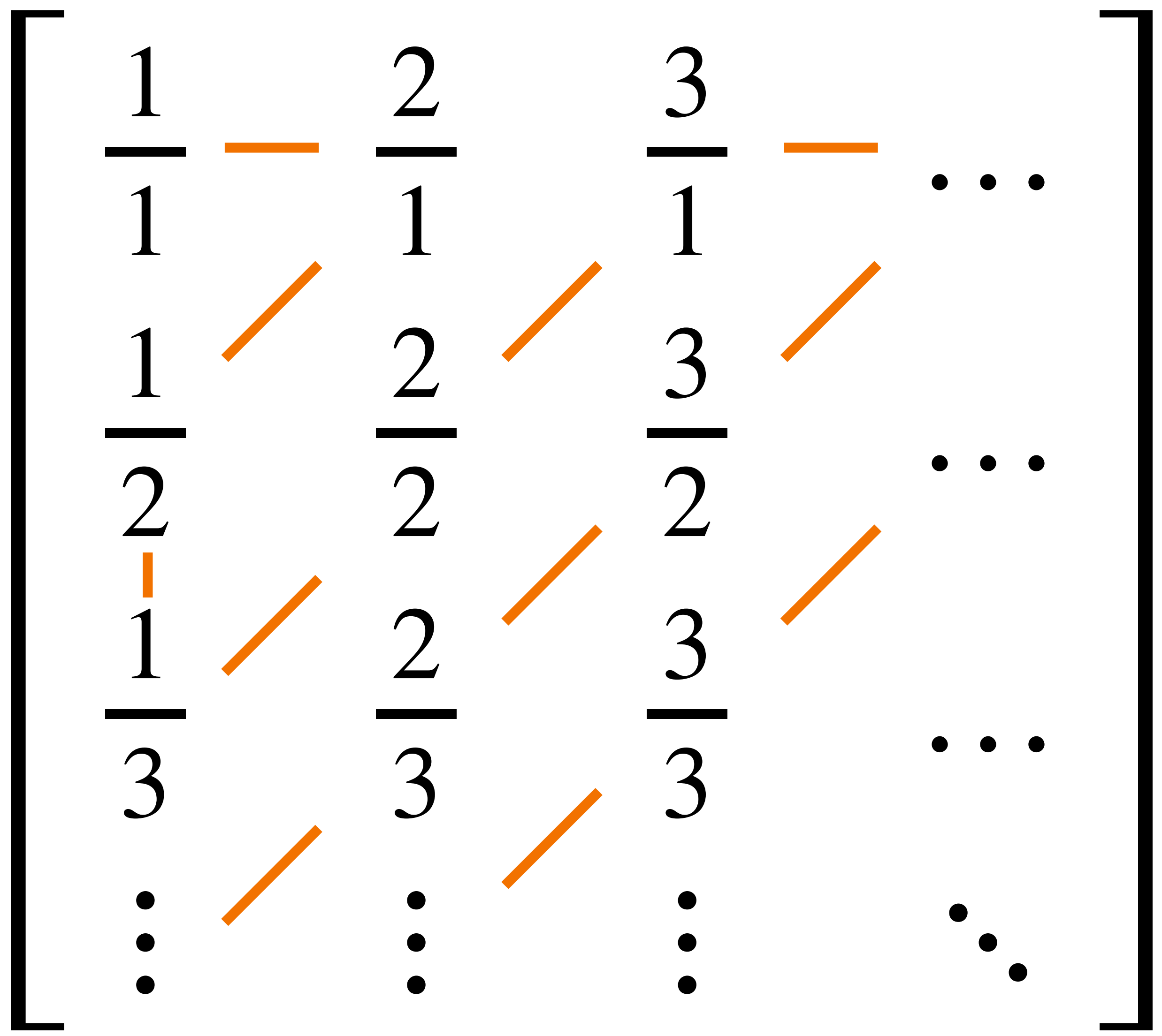
Henri Poincaré

⭐⭐⭐⭐⭐

- “No one would expel us from the paradise that Georg Cantor has opened for us”

David Hilbert

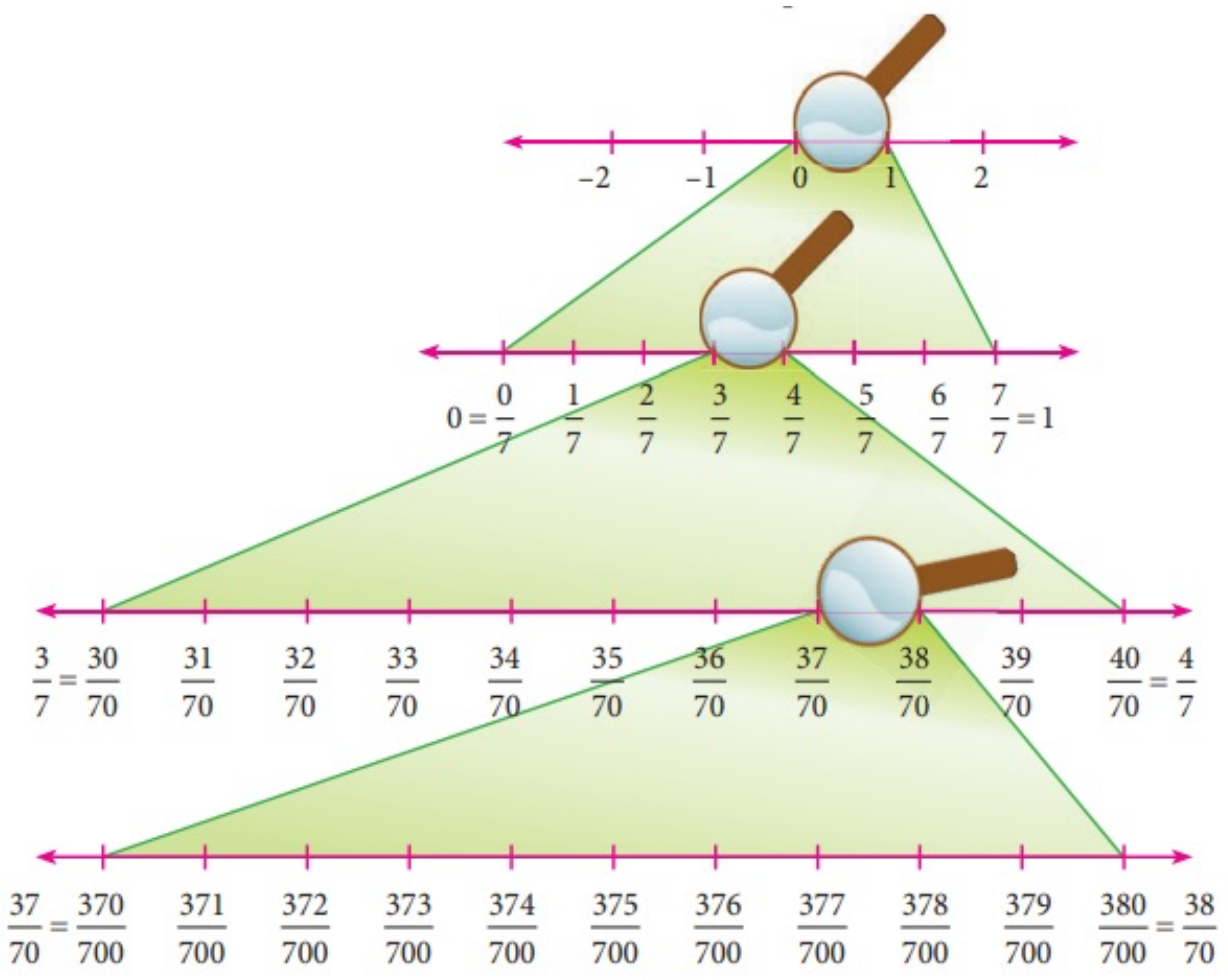
Rational numbers are countable



- Cantor showed there are as many rational numbers as whole numbers
- Cantor's diagonalization proof

ℝ

- ℝ is infinitely dense
- Given a number there is no "next number"
- How much more than ℕ or ℤ?



$\mathbb{R}$

Real numbers are not countable

Proof:

- Start with a list of numbers
- We can construct an element that was not in the list
- Thus, the set is uncountable

$$s_1 = 0.91476382934872365 9$$
$$s_2 = 0.14123985647312454 2$$
$$s_3 = 0.42564376274859842 7$$
$$s_4 = 0.51281238964672123 7$$
$$s_5 = 0.73564212371467213 8$$
$$s_6 = 0.83649111238964178 3$$
$$s_7 = 0.93658370123487123 1$$

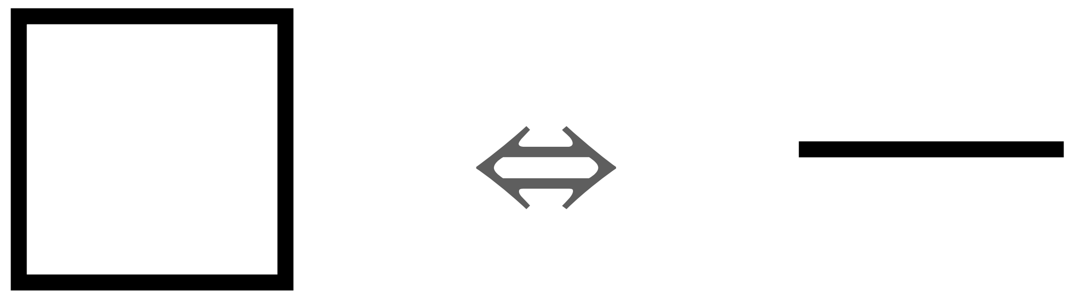
$$\vdots$$

$$s_? = 0.0569321\ldots$$

Cantor showed that there are different orders of infinity

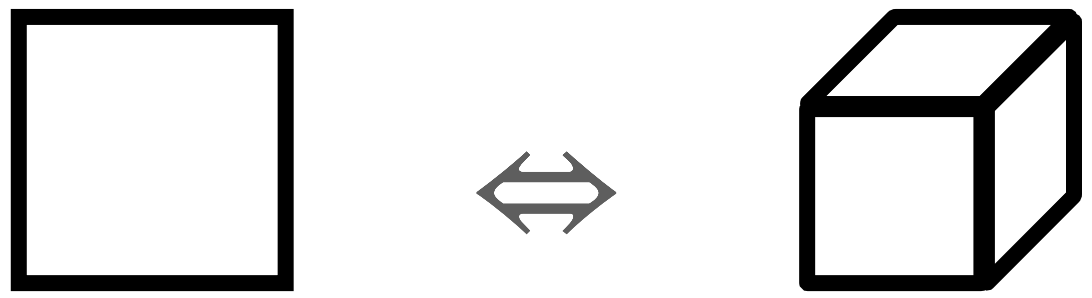
“Je le vois, mais je ne le crois pas!”

- Consider a unit line segment and a unit square.

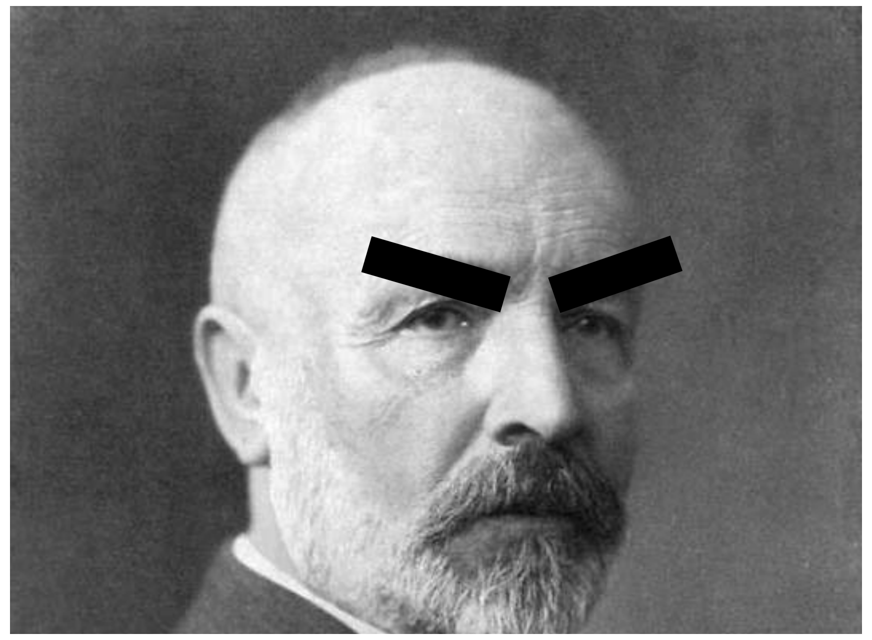


- There exists a 1-to-1 correspondence between each point on the line segment and each point in the square

- Moreover, for each n, there exists a correspondence between the points on the line segment and all the points in the n-dimensional cube



Virulent opposition



- Cantor initially tried to befriend Kronecker, by inviting him to his mountain resort, but they did not like each other's perspective
- Kronecker actively stalled the publication on the irrelevance of dimension
- In 1883, Cantor decides to apply for a professorship in Berlin, well knowing that it would not be possible, just to annoy Kronecker
- Kronecker retaliated by pretending to be trying to publish in *Acta Mathematica*, which caused Cantor to write to Mittag-Leffler
- This feud took a toll on his health

Transfinite numbers

- The cardinality of a set is the measure of the number of elements it contains
- For infinite sets, Cantor denoted the transfinite cardinal number with $\aleph$
- The lowest order of infinity (of the integers and the rationals) was named $\aleph_0$, and he believed that the alephs continued

$$S = \{1,2,3\} \rightarrow |S| = 3$$

$$Q = \{Apples, Dogs, Key, Computer\} \rightarrow |Q| = 4$$

$$|\mathbb{N}| = \aleph_0$$

$$\aleph_0, \aleph_1, \aleph_2, \dots$$

$$\aleph_0 + 1 = \aleph_0$$

Adding one element to an infinite set

$$\aleph_0 + \aleph_0 = \aleph_0$$

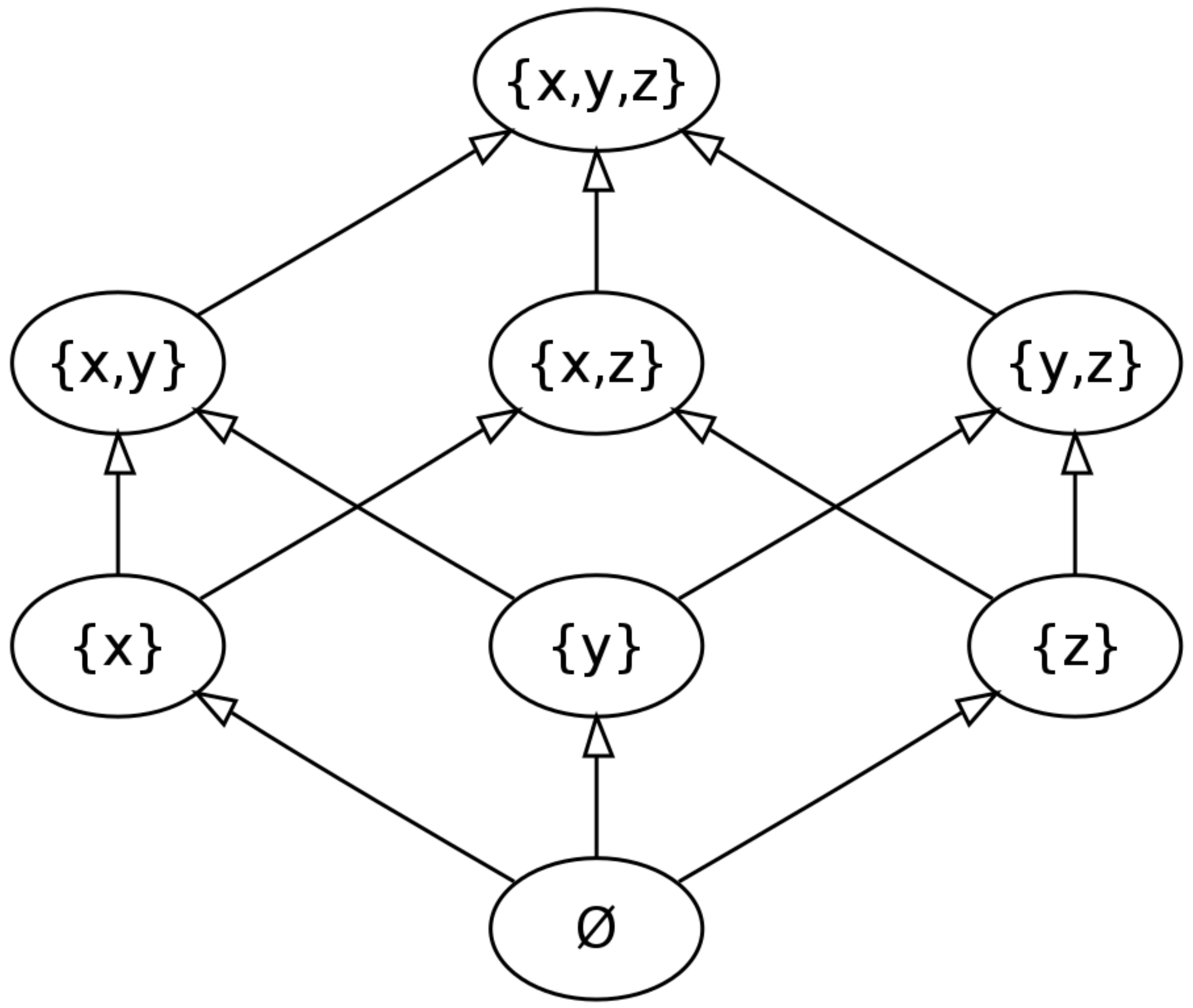
Adding infinite elements to an infinite set

$$\aleph_0 \times \aleph_0 = \aleph_0$$

Combining two infinite sets

Does not affect the order of infinity

The continuum hypothesis



Given a set Ω, its power set is called 2^{Ω} and consists of the set of all subsets of Ω.

$$\Omega = \{1,2,3\}$$

$$2^{\Omega} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

The power set operation can return something with bigger cardinality than the original set

	S1	**S2**	**S3**	**S4**	**S5**	**S6**	**...**
0	1	1	0	1	0	0	...
1	0	1	1	1	1	1	...
2	0	0	0	0	1	1	...
3	1	0	0	1	0	1	...
4	0	0	0	0	0	0	...
5	0	1	0	1	1	1	...
...	...	...	...	...	...	...	...

S1
0
0
1
0
1
0
...

The continuum hypothesis

Cantor knew that the cardinal number of the continuum $\mathfrak{c}$ consisted of all possible subsets of the set of integers

$$\mathfrak{c} = 2^{\aleph_0}$$

He was wondering if there is any other order between $\mathfrak{c}$ and $\aleph_0$

And, he wanted to prove that:

$$\aleph_1 = 2^{\aleph_0}$$

$$\nexists S : \aleph_0 < |S| < 2^{\aleph_0}$$

There is no set whose cardinality is strictly between that of the integers and the real numbers

A.K.A. "The continuum hypothesis"

Shakespeare and mental illness

- After having stated this problem, he worked on solving it, and he sent over an extended period letters to Mittag-Leffler about a proof of its truth or its falseness
- He then “slowly went mad”, suffering from recurring nervous breakdowns
- While recovering, he became a Shakespearean scholar
- He wanted to prove that the true author of Shakespeare’s plays was Francis Bacon
- He saw its hypothesis as a dogma, meanwhile other mathematicians started to realise the power of his work

David Hilbert’s Problems

Top 23 unsolved problems in 1900

1. Cantor's problem of the cardinal number of the continuum.

2. The compatibility of the arithmetical axioms.

3. The equality of the volumes of two tetrahedra of equal bases and equal altitudes.

4. Problem of the straight line as the shortest distance between two points.

5. Lie's concept of a continuous group of transformations without the assumption of the differentiability of the functions defining the group.

6. Mathematical treatment of the axioms of physics.

7. Irrationality and transcendence of certain numbers.

8. Problems of prime numbers (The "Riemann Hypothesis”).

…

“Funny” anecdote

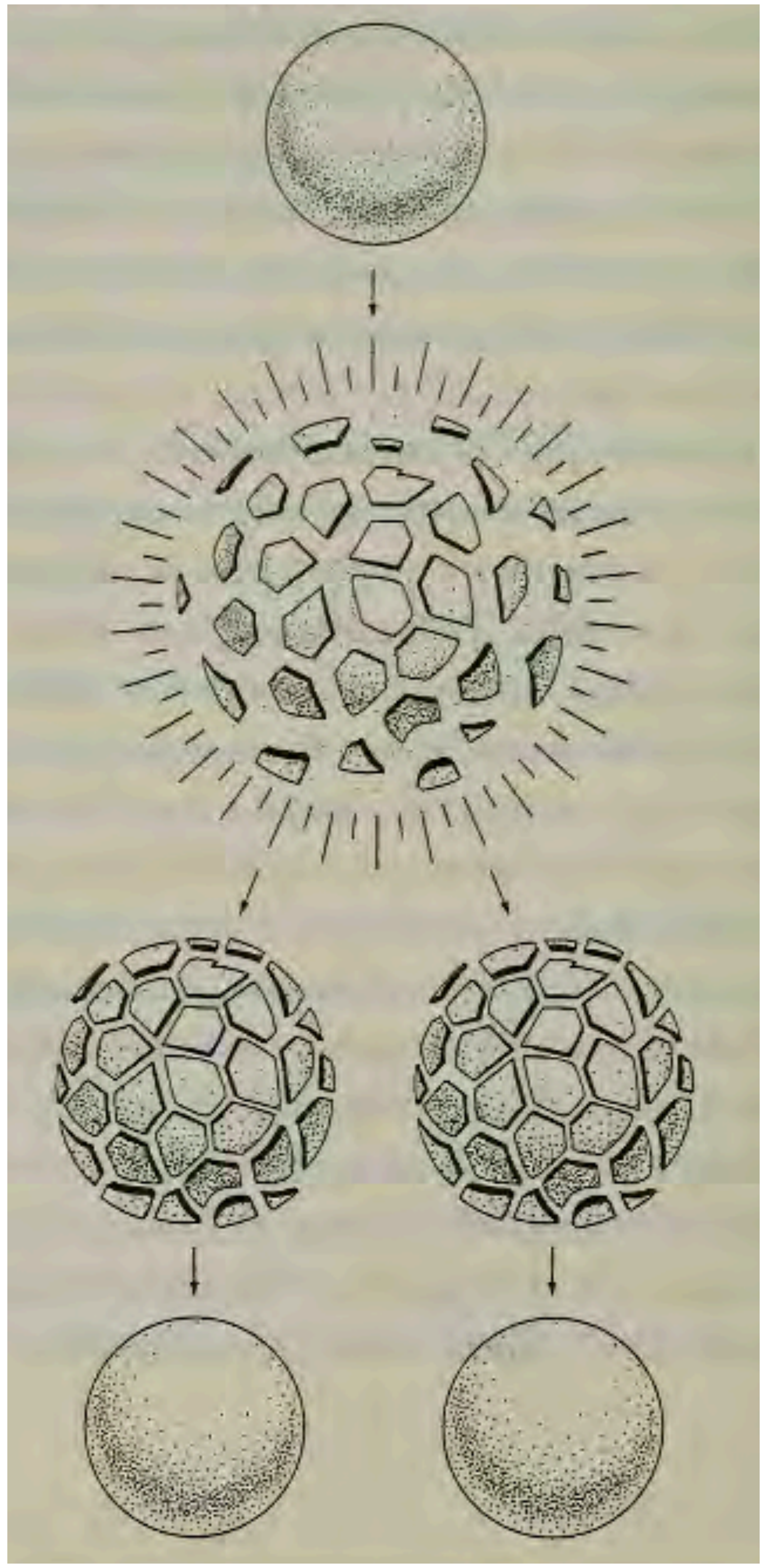
- In 1908, Cantor promised a British mathematician to send a mathematical paper to the Journal of the London Mathematical Society, but never did so. However, he had an invitation open and in Sept. 1911 he came to St.Andrews University as an invited foreign scholar.
- The invitation was from the department of mathematics, expecting a discussion of set theory and infinity.
- He instead gave a presentation on the Bacon-Shakespeare issue and then immediately left for London, where he started writing letters to Bertrand Russell, containing “words continuing into the margins and lines written from top to bottom across lines written from left to right”
- Russell described Cantor as one of the greatest intellects of the 19th century but “no one will be surprised to learn that he spent a large part of his life in a lunatic asylum”

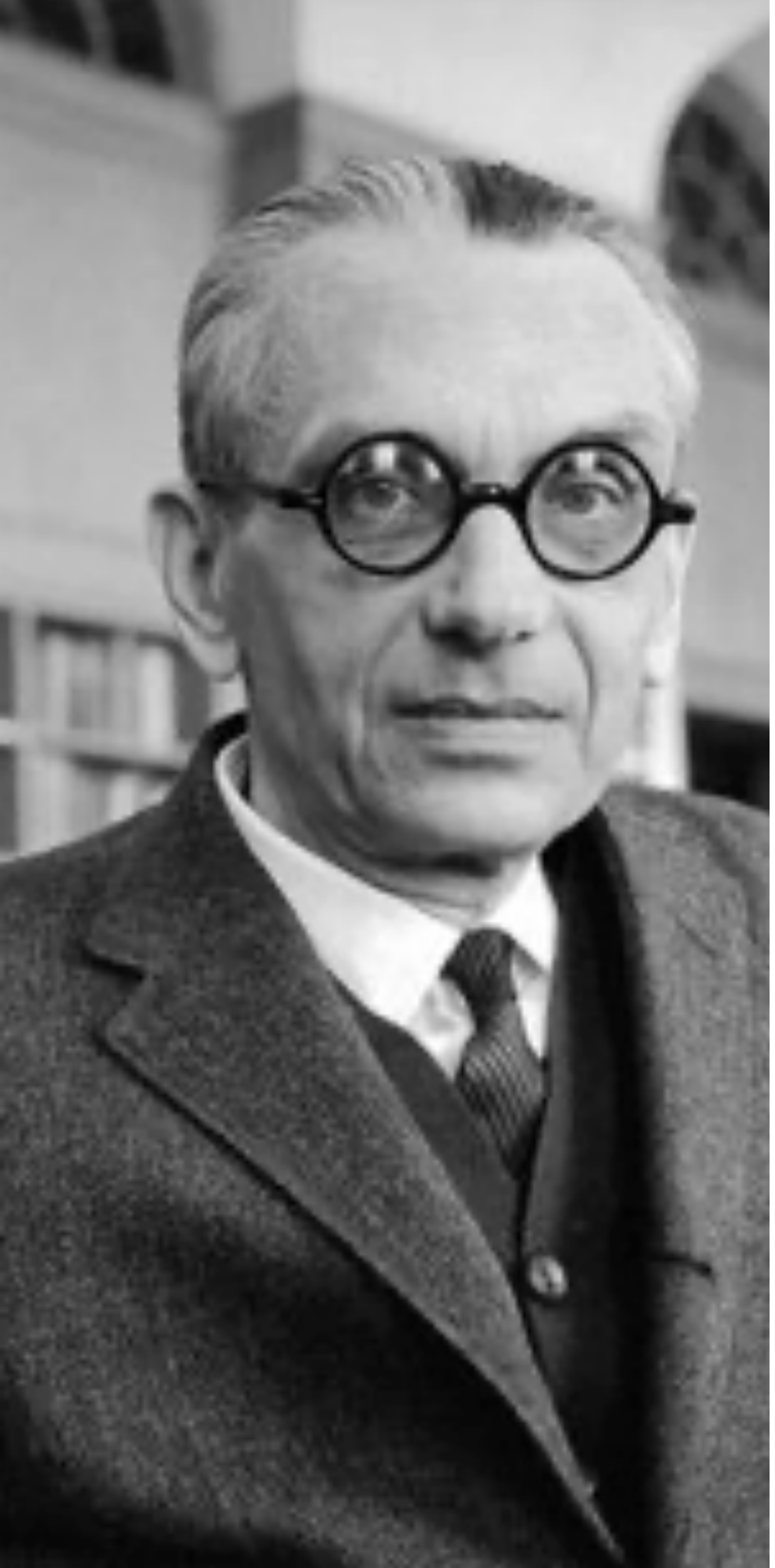
Axiom of choice and Russel’s paradox

- The continuum hypothesis requires the introduction of the so called “axiom of choice”
- Which, in turn, allows the existence of the Banach-Tarski paradox
    - A sphere in euclidean space can be decomposed into a finite number of parts and reassembled to form two spheres each with the same radius as the original one

https://www.youtube.com/watch?v=WaxiL4IRIYw

https://www.youtube.com/watch?v=s86-Z-CbaHA





An answer?

- It is possible to extend set theory to include the continuum hypothesis (and the axiom of choice), without causing contradictions

Kurt Gödel,1940

- It is possible to extend set theory to include the opposite of the continuum hypothesis (and the axiom of choice) without causing contradictions

Paul Cohen, 1963

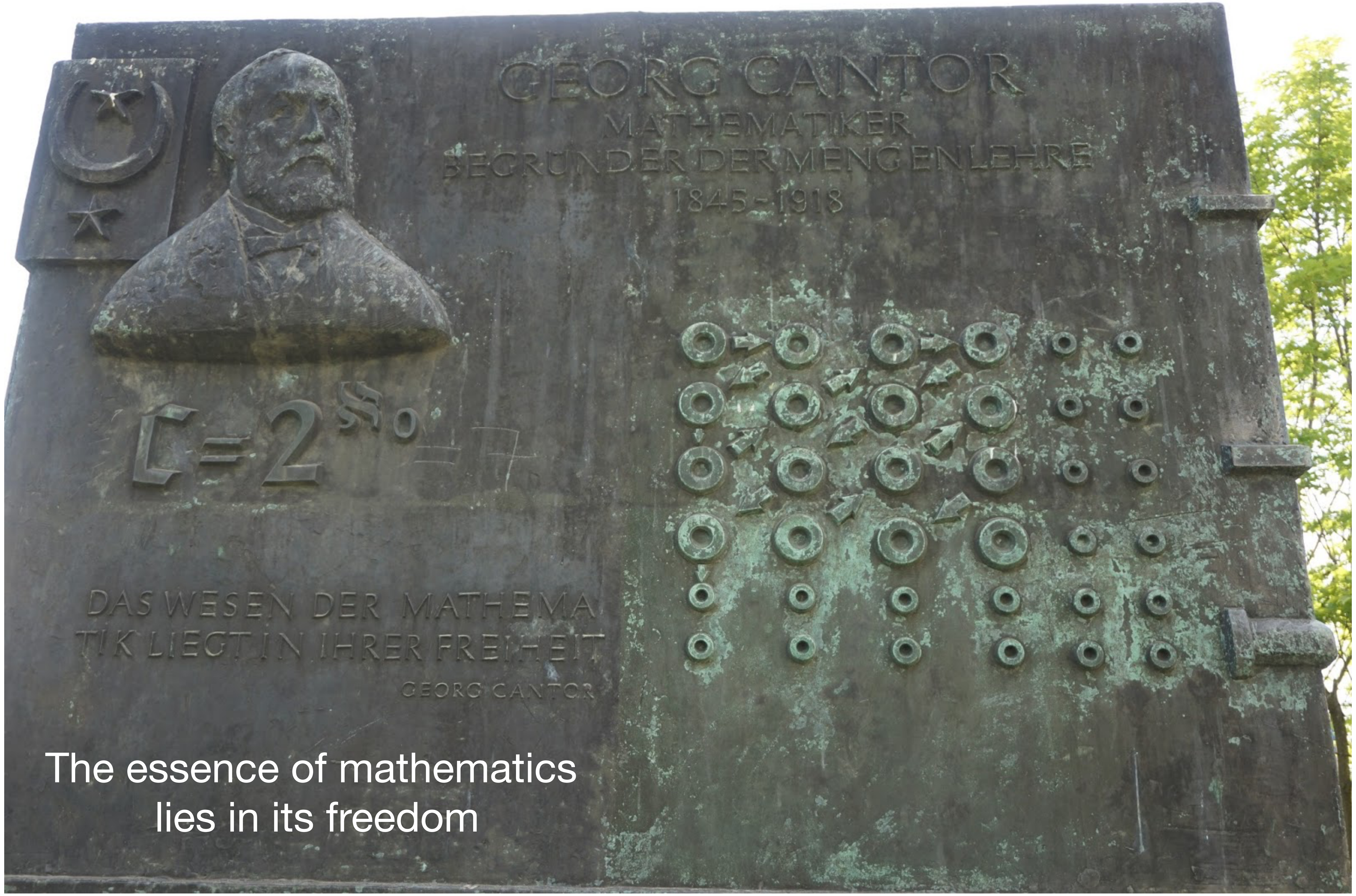
The continuum hypothesis cannot be proven

The infinite brightness of the Chaluk

- Gödel and Cohen told us there are some truths beyond our reach
- Does infinity exist?
  - How big is out universe? Can space be infinitesimally divided?
- Do numbers exist? Does the continuum exist?

?

Thanks for your attention!



The essence of mathematics
lies in its freedom